

Optimal Control Theory. Introduction

Contents

1	Classical control vs. optimal control	2
2	Problem Formulation	3
2.1	The Mathematical Model	3
2.2	Physical Constraints	7
2.3	The Performance Measure	8
2.3.1	Minimum-Time Problems	9
2.3.2	Terminal Control Problems	10
2.3.3	Minimum energy problems. Minimum-Control-Effort Problems	11
2.3.4	Tracking problems	12
2.3.5	Regulator problems	13
2.4	The Optimal Control Problem	13
2.4.1	Form of an optimal control	14
2.5	Examples and exercises	14

Classical control vs. optimal control

Classical control system design is generally a trial-and-error process in which various methods of analysis are used iteratively to determine the design parameters of an "acceptable" system. Acceptable performance is generally defined in terms of time and frequency domain criteria such as rise time, settling time, peak overshoot, gain and phase margin, and bandwidth.

Radically different performance criteria must be satisfied, however, by the complex, multiple-input, multiple-output systems required to meet the demands of modern technology.

For example, the design of a spacecraft attitude control system that minimizes fuel expenditure is not amenable to solution by classical methods. A new and direct approach to the synthesis of these complex systems, called *optimal control theory*, has been made feasible by the development of the digital computer. The objective of optimal control theory is *to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance criterion*. Later, we shall give a more explicit mathematical statement of "the optimal control problem," but first let us consider the matter of problem formulation, (Kirk, 2004).

Problem Formulation

The axiom "A problem well put is a problem half solved" may be a slight exaggeration, but its intent is nonetheless appropriate. In this section, we shall review the important aspects of problem formulation, and introduce the notation and nomenclature to be used in the following chapters, (Kirk, 2004).

The formulation of an optimal control problem requires, (Kirk, 2004):

- A mathematical description (or model) of the process to be controlled.
- A statement of the physical constraints
- Specification of a performance criterion

2.1 The Mathematical Model

Modeling the process is a nontrivial part of any control problem. In the followings the systems will be described by ordinary differential equations, in **state variable** form.

If we denote by:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{bmatrix}$$

the *state variable vector* (or the *state vector*) of the system at time t , and by

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2 \\ \dots \\ u_m(t) \end{bmatrix}$$

the *control vector* to the system at time t , then the process may be described by n first-order differential equations:

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \\ \dot{x}_2(t) &= f_2(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \\ &\dots \\ \dot{x}_n(t) &= f_n(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t) \end{aligned} \quad (2.1)$$

The state equation can be the written in a vector form as:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (2.2)$$

Example 2.1.1 , from (Kirk, 2004).

The car shown parked in Figure 2.1 is to be driven in a straight line away from point 0. The distance of the car from point 0 at time t is denoted by $d(t)$. To simplify the model, let us approximate the car by a unit point mass



Figure 2.1: A simplified control problem

that can be accelerated by using the throttle or decelerated by using the brake. If the position of the car at any time moment is $d(t)$, then the first derivative $\dot{d}(t)$ is the velocity of the car and the second derivative $\ddot{d}(t)$ is the acceleration, that will give the following differential equation:

$$\ddot{d}(t) = \alpha(t) + \beta(t)$$

where the control α is throttle acceleration and β is braking deceleration. Selecting the position and velocity as state variables, that is:

$$\begin{aligned} x_1(t) &= d(t) \\ x_2(t) &= \dot{d}(t) \end{aligned}$$

and letting the control variables:

$$u_1(t) = \alpha(t) \text{ and; } u_2(t) = \beta(t)$$

we find that the state equations become:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u_1(t) + u_2(t) \end{aligned} \quad (2.3)$$

or using matrix notation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}(t)$$

This is the mathematical model of the process in the state form.

Definition 2.1.1 For the system be described by equation: $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$, a history of control values during the interval $[t_0, t_f]$ is denoted by $\mathbf{u}$ and is called a control history or simply a control.

Definition 2.1.2 For the system be described by equation: $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$, a history of state values in the interval $[t_0, t_f]$ is called a state trajectory and is denoted by $\mathbf{x}$.

Exercise 2.1.1 Write a set of state equations for the spring-mass-damper system shown in Figure 2.2. The applied force is $r(t)$, the block has mass M , the spring constant is k and the coefficient of viscous friction is f . For this

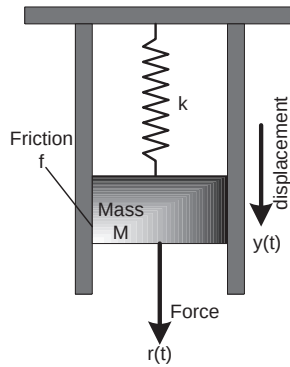


Figure 2.2: Spring-mass-damper system

system, Newton second law of motion states that:

$$M \frac{d^2 y(t)}{dt^2} + f \frac{dy(t)}{dt} + k y(t) = r(t)$$

Exercise 2.1.2 Conic tank. Consider the conic tank shown in Figure 2.3. The volume of the liquid in the tank is equal to $kh(t)^3$, where k is a constant and h , the height of the liquid. The liquid is filled in the tank at a flow-rate q_e and discharged at a flow-rate q_s .

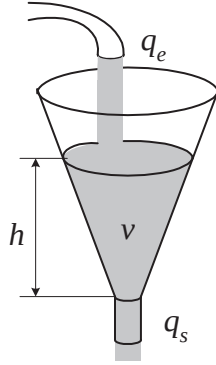


Figure 2.3: Conic tank

Obtain the state equations which describe this system.

Hint. The variables for this system are: the volume of liquid $v(t)$, the height of the liquid $h(t)$, the input flow-rate $q_e(t)$ and the output flow-rate $q_s(t)$. From the statement of the problem, the relation between the volume and the height is:

$$v(t) = kh(t)^3$$

Two equations can be found in the physical laws which govern the system. The volume of liquid in the tank is increased because of the difference between the input and output flow-rates, so:

$$q_e(t) - q_s(t) = \frac{dv(t)}{dt}$$

A relation between the height of the liquid and the output flow-rate obtained from Toricelli's theorem in ideal conditions is:

$$q_s(t) = CS\sqrt{2gh(t)} = B\sqrt{h(t)}$$

where C is the contraction coefficient and S the area of the outlet $[m^2]$ and $B = CS\sqrt{2g}$ is a constant.

The state equations can be formulated in terms of the state variable $x(t) = h(t)$, input $u(t) = q_e(t)$ and output $y(t) = q_s(t)$.

2.2 Physical Constraints

After a mathematical model has been selected, the next step is to define the physical constraints on the state and control values. To illustrate some typical constraints, let us return to the automobile whose model was determined in Example 2.1.1.

Example 2.2.1 , from (Kirk, 2004).

Consider the problem of driving the car in Figure 2.4 between the points 0 and d_{final} . Assume that the car starts from rest and stops upon reaching point d_{final} .



Figure 2.4: A simplified control problem

First let us define the state constraints. If t_0 is the time of leaving 0, and t_f is the time of arrival at d_{final} , then:

$$\begin{aligned} x_1(t_0) &= 0 \\ x_1(t_f) &= d_{final} \end{aligned} \tag{2.4}$$

In addition, since the automobile starts from rest and stops at d_{final} , the initial and final velocities of the car are zero:

$$\begin{aligned} x_2(t_0) &= 0 \\ x_2(t_f) &= 0 \end{aligned} \tag{2.5}$$

In matrix notation, these boundary conditions are:

$$\mathbf{x}(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0} \text{ and } \mathbf{x}(t_f) = \begin{bmatrix} d_{final} \\ 0 \end{bmatrix} = \mathbf{0} \tag{2.6}$$

If we assume that the car does not back up, then the additional constraints

$$\begin{aligned} 0 &\leq x_1(t) \leq d_{final} \\ 0 &\leq x_2(t) \end{aligned} \tag{2.7}$$

are also imposed.

What are the constraints on the control inputs (acceleration)? We know that the acceleration is bounded by some upper limit which depends on the

capability of the engine and that the maximum deceleration is limited by the braking system parameters. If the maximum acceleration is $M_1 > 0$, and the maximum deceleration is $M_2 > 0$, then the controls must satisfy

$$\begin{aligned} 0 &\leq u_1(t) \leq M_1 \\ -M_2 &\leq u_2(t) \leq 0 \end{aligned} \tag{2.8}$$

In addition, if the car starts with G gallons of gas and there are no service stations on the way, another constraint can be:

$$\int_{t_0}^{t_f} [k_1 u_1(t) + k_2 x_2(t)] dt \leq G \tag{2.9}$$

which assumes that the rate of gas consumption is proportional to both acceleration and speed with constants of proportionality k_1 and k_2 , (Kirk, 2004).

Definition 2.2.1 A control history which satisfies the control constraints during the entire time interval $[t_0, t_f]$ is called an *admissible control*.

Definition 2.2.2 A state trajectory which satisfies the state variable constraints during the entire time interval $[t_0, t_f]$ is called an *admissible trajectory*.

Admissibility is an important concept, because it reduces the range of values that can be assumed by the states and controls. Rather than consider all control histories and their trajectories to see which are the best (according to some criterion) we investigate only those trajectories and controls that are admissible, (Kirk, 2004).

2.3 The Performance Measure

Having already considered the modeling of systems and the determination of state and control constraints, the next step in formulating an optimal control problem is obtaining a performance measure. This is selected by the designer in order to evaluate the performance of a system quantitatively.

An *optimal control* is defined as one that *minimizes* (or *maximizes*) the performance measure. The terms *performance measure*, *performance index* or *cost function* can be used interchangeably in the followings.

For example, the statement "transfer from point A to point B as quickly as possible" clearly indicates that elapsed time is the performance measure to be minimized.

On the other hand, the statement "maintain the position and velocity of the system near zero with small expenditure of control energy", does not instantly suggest a unique performance measure. In such problems the designer may be required to try several performance measures before selecting one which yields what he considers to be optimal performance, (Kirk, 2004).

Example 2.3.1 , from (Kirk, 2004).

Consider the automobile problem begun in Example 2.1.1. The state equations and physical constraints have been defined; now we turn to the selection of a performance measure. Suppose the objective is to make the car reach the point e as quickly as possible, then the performance measure J is given by:

$$J = t_f - t_0$$

In all that follows it will be assumed that the performance of a system is evaluated by a measure of the form:

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (2.10)$$

where t_0 and t_f are the initial and final time, h and g are scalar functions. t_f may be specified or "free" (unspecified), depending on the problem statement.

Starting from the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$, and applying a control signal $\mathbf{u}(t)$, for $t \in [t_0, t_f]$, causes a system to follow some state trajectory; the performance measure assigns a unique real number to each trajectory of the system.

2.3.1 Minimum-Time Problems

Problem: To transfer a system from an arbitrary initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to a specified final state $\mathbf{x}(t_f) = \mathbf{x}_f$ in the shortest possible time.

The performance measure to be minimized is:

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt \quad (2.11)$$

The automobile example discussed in Ex 2.3.1 is a minimum time problem, (Kirk, 2004).

2.3.2 Terminal Control Problems

Problem: To minimize the deviation of the final state of a system from its desired value $\mathbf{r}_f$, where $\mathbf{r}_f = [r_{1f} \ r_{2f} \ \cdots \ r_{nf}]^T$.

A possible performance measure is:

$$J = \sum_{i=1}^n [x_i(t_f) - r_{if}]^2 \quad (2.12)$$

The differences between x_i and r_{if} are squared because positive and negative deviations are usually undesirable. An alternative would be using absolute values, but the quadratic form in Eq. (2.12) is easier to handle mathematically.

Using matrix notation, we have:

$$J = [\mathbf{x}(t_f) - \mathbf{r}_f]^T [\mathbf{x}(t_f) - \mathbf{r}_f] \quad (2.13)$$

or this can be written as:

$$J = \|\mathbf{x}(t_f) - \mathbf{r}_f\|^2 \quad (2.14)$$

$\|\mathbf{x}(t_f) - \mathbf{r}_f\|$ is called the *norm* of vector $[\mathbf{x}(t_f) - \mathbf{r}_f]$.

To allow greater generality, we can insert a real symmetric positive semi-definite $n \times n$ matrix $\mathbf{H}$ to obtain:

$$J = [\mathbf{x}(t_f) - \mathbf{r}_f]^T \mathbf{H} [\mathbf{x}(t_f) - \mathbf{r}_f] \quad (2.15)$$

If $\mathbf{H}$ is the identity matrix (2.13) and (2.15) are identical.

Suppose that $\mathbf{H}$ is a diagonal matrix. The assumption that $\mathbf{H}$ is positive semi-definite implies that all of the diagonal elements are nonnegative. By adjusting the element values we can weight the relative importance of the deviation of each of the states from their desired values. Thus, by increasing h_{ii} (the ii element of $\mathbf{H}$) we attach more significance to deviation of $x_i(t_f)$ from its desired value; by making h_{ii} zero we indicate that the final value of x_i is of no concern whatsoever, (Kirk, 2004).

Example 2.3.2 *For a second order system, a possible performance measure that aims to minimize the deviation of the final, states $x_1(t_f)$ and $x_2(t_f)$ from the desired values $r_{1f} = 1$ and $r_{2f} = 3$, respectively, will be:*

$$J = [x_1(t_f) - 1]^2 + [x_2(t_f) - 3]^2$$

If it is more important that the first final state is closer to 1 than the second final state is closer to 3, we may attach weights to each term in J , for example:

$$J = 100 \cdot [x_1(t_f) - 1]^2 + 0.1 \cdot [x_2(t_f) - 3]^2$$

In this case, $\mathbf{H}$ is a diagonal matrix, equal to:

$$\mathbf{H} = \begin{bmatrix} 100 & 0 \\ 0 & 0.1 \end{bmatrix}$$

2.3.3 Minimum energy problems. Minimum-Control-Effort Problems

Problem: To find an admissible control that drives a system on an admissible state trajectory with a minimum expenditure of control effort.

The meaning of the term "minimum control effort" depends upon particular physical application; therefore the performance measure may assume various forms.

The most common terms to be found in this type of performance measure are functions of the squares of the control inputs.

Squares are used partly to ensure that the resulting expressions are never negative (so that finding minima over time is not complicated by sign change); and partly because the square of a signal variable is often an indication of some kind of *energy* transfer. Thus, minimizing the square minimizes the energy expenditure - and hence cost, (Dutton et al., 1997).

Example 2.3.3 , from (Dutton et al., 1997).

For example, if a system has a voltage input signal $v(t)$ feeding into a constant resistance, then the energy expended during control is given by (power dissipated) $\times$ (time), namely:

$$\frac{v^2(t)}{R} \cdot t$$

If $v^2(t)$ is integrated with respect to time, a result having units of V^2s is obtained (that is, the area under $v^2(t)$ vs. time graph). Minimizing such an integral, so long as R is constant, will therefore minimize energy expenditure.

Similarly, in a mechanical system, if $x(t)$ represents a velocity, then its inclusion as $x^2(t)$ in an integral performance measure will seek to minimize instantaneous kinetic energy, so long as the mass of the moving object is constant.

For a general control signal $u(t)$, as shown in figure 2.5, a suitable term for use as a performance measure (J) might therefore be

$$J = \int_{t_0}^{t_f} u^2(t) dt$$

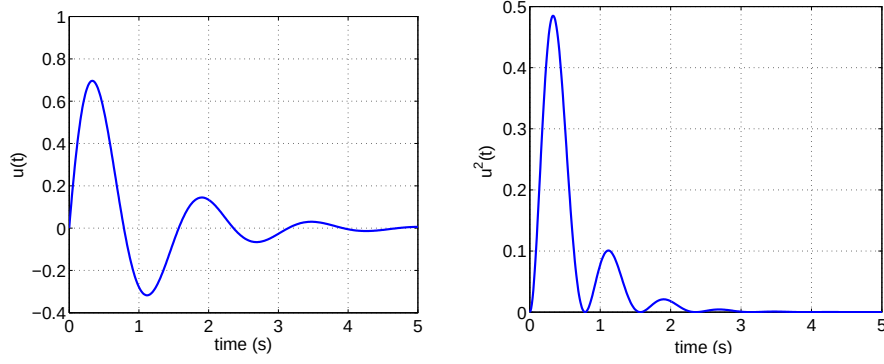


Figure 2.5: (left) A control signal $u(t)$ vs. time (right) $u^2(t)$ vs. time

where the integral is taken over some time interval of interest, and the value of J will be the area under the graph in figure 2.5 (right). As before, if $u(t)$ represents for example a voltage or a current signal, then minimizing J will minimize energy expenditure.

For several control inputs the general form of performance measure is:

$$J = \int_{t_0}^{t_f} [\mathbf{u}^T(t) \mathbf{R} \mathbf{u}(t)] dt = \int_{t_0}^{t_f} \|\mathbf{u}(t)\|_R^2 dt \quad (2.16)$$

where $\mathbf{R}$ is a real symmetric positive definite weighting matrix. The elements of $\mathbf{R}$ may be functions of time if it is desired to vary the weighting on control-effort expenditure during the interval $[t_0, t_f]$.

Obs. In the following sections we will denote by a subscript $\mathbf{S}$ to the squared norm of a vector, the product:

$$\|\mathbf{x}\|_S^2 = \mathbf{x}^T \mathbf{S} \mathbf{x}$$

2.3.4 Tracking problems

Problem: To maintain the system state $\mathbf{x}(t)$ as close as possible to the desired state $\mathbf{r}(t)$ in the interval $[t_0, t_f]$.

As a performance measure we select

$$J = \int_{t_0}^{t_f} \|\mathbf{x}(t) - \mathbf{r}(t)\|_{Q(t)}^2 dt, \quad (2.17)$$

where $\mathbf{Q}(t)$ is a real symmetric $n \times n$ matrix that is positive semi-definite for all $t \in [t_0, t_f]$. The elements in matrix $\mathbf{Q}$ are selected to weight the

relative importance of the different components of the state vector and to normalize the numerical values of the deviations. For example if $\mathbf{Q}$ is a constant diagonal matrix and q_{ii} is zero, this indicates that deviations of x_i are of no concern, (Kirk, 2004).

If the set of admissible controls is bounded, e.g. $|u_i(t)| \leq 1, i = 1, 2, \dots, m$, then Eq (2.17) is a reasonable performance measure. If the controls are not bounded, minimizing (2.17) results in controls with impulses and their derivatives. To avoid placing bounds on the admissible controls, or if control energy is to be conserved, we use the modified performance measure

$$J = \int_{t_0}^{t_f} [\|\mathbf{x}(t) - \mathbf{r}(t)\|_{\mathbf{Q}(t)}^2 + \|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2] dt, \quad (2.18)$$

$\mathbf{R}(t)$ is a real symmetric positive definite $m \times m$ matrix for all $t \in [t_0, t_f]$.

It may be especially important that the states be close to their desired values at the final time. In this case, the performance measure

$$J = \|\mathbf{x}(t_f) - \mathbf{r}(t_f)\|_{\mathbf{H}}^2 + \int_{t_0}^{t_f} [\|\mathbf{x}(t) - \mathbf{r}(t)\|_{\mathbf{Q}(t)}^2 + \|\mathbf{u}(t)\|_{\mathbf{R}(t)}^2] dt, \quad (2.19)$$

could be used. $\mathbf{H}$ is a real symmetric positive semi-definite $n \times n$ matrix, (Kirk, 2004).

2.3.5 Regulator problems

A regulator problem is the special case of a tracking problem which results when the desired state values are zero ($\mathbf{r}(t) = 0$ for all $t \in [t_0, t_f]$), for example:

$$J = \int_{t_0}^{t_f} \|\mathbf{x}(t)\|_{\mathbf{Q}(t)}^2 dt, \quad (2.20)$$

2.4 The Optimal Control Problem

The theory developed in the subsequent sections is aimed at solving the following problem:

Find an *admissible control* $\mathbf{u}^*$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), \mathbf{u}(t), t)$$

to follow an *admissible trajectory* $\mathbf{x}^*$ that *minimizes* the performance

measure:

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt$$

$\mathbf{u}^*$ is called an *optimal control* and $\mathbf{x}^*$ and *optimal trajectory*.

2.4.1 Form of an optimal control

Definition 2.4.1 , (Kirk, 2004)

If a functional relationship of the form

$$\mathbf{u}^*(t) = \mathbf{f}(\mathbf{x}(t), t) \quad (2.21)$$

can be found for the optimal control at time t , then the function $\mathbf{f}$ is called the optimal control law, or the optimal policy.

Notice that Eq. (2.21) implies that $\mathbf{f}$ is a rule which determines the optimal control at time t for any admissible state value at time t .

For example, if

$$\mathbf{u}^*(t) = \mathbf{F}\mathbf{x}(t)$$

where $\mathbf{F}$ is an $m \times n$ matrix of real constants, then we would say that the optimal control law is linear, time-invariant feedback of the states.

Definition 2.4.2 , (Kirk, 2004).

If the optimal control is determined as a function of time for a specified initial state value, that is:

$$\mathbf{u}^*(t) = \mathbf{e}(\mathbf{x}(t_0), t)$$

then the optimal control is said to be in the open-loop form.

Conceptually, it is helpful to imagine the difference between an optimal control law and an open-loop optimal control as shown in Figure 2.6. Although engineers may prefer closed-loop solutions to optimal control problems there are cases when an open-loop control may be feasible. For example, in the radar tracking of a satellite, once the orbit is set, little can happen to cause an undesired change in the trajectory parameters.

2.5 Examples and exercises

Example 2.5.1 (Kirk, 2004) *Figure 2.7 shows a rocket that is to be approximated by a particle of instantaneous mass $m(t)$. The instantaneous velocity*

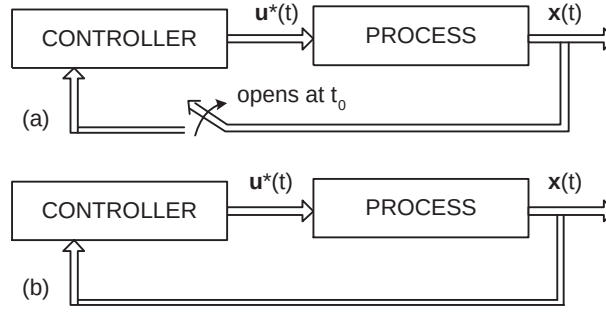


Figure 2.6: (a) Open-loop optimal control. (b) Optimal control law, (Kirk, 2004)

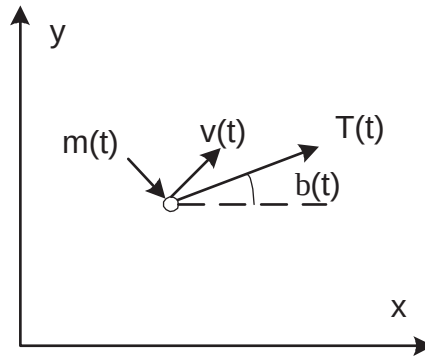


Figure 2.7: Rocket

is $v(t)$. $T(t)$ is the thrust and $b(t)$ is the thrust angle. If we assume no aerodynamic or gravitational forces, and if we select $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$, $x_5 = m$, $u_1 = T$, $u_2 = b$, the state equations are:

$$\begin{aligned}
 \dot{x}_1(t) &= x_2(t) \\
 \dot{x}_2(t) &= \frac{[u_1(t) \cos u_2(t)]}{x_5(t)} \\
 \dot{x}_3(t) &= x_4(t) \\
 \dot{x}_4(t) &= \frac{[u_1(t) \sin u_2(t)]}{x_5(t)} \\
 \dot{x}_5(t) &= -c \cdot x_5(t)
 \end{aligned}$$

where c is a constant. The rocket starts from rest at the point $x = 0$, $y = 0$.

(a) Determine a set of physically reasonable state and control constraints.

- (b) *Suggest a performance measure, and any additional constraints imposed, if the objective is to make $y(t_f) = 3\text{mi}$ and maximize $x(t_f)$; t_f is imposed.*
- (c) *Suggest a performance measure, and any additional constraints imposed if it is desired to reach the point $x = 500\text{ mi}$, $y = 3\text{ mi}$ in 2.5 min with maximum possible vehicle mass.*

Solution

- (a) *A set of physically reasonable state constraints could be imposed as:*

- *The coordinate $x(t)$ must be positive*

$$x(t) \geq 0$$

- *The vehicle mass physically varies between a maximum value and a minimum one (the mass could depend on the mass of fuel in the tanks). Then:*

$$M_{\min} \leq x_5(t) \leq M_{\max}$$

A set of physically reasonable control constraints could be imposed as:

- *The vehicle thrust has normally a maximum value given by the engine capability.*

$$0 \leq u_1 \leq T_{\max}$$

- *If the vehicle is supposed to move only in the positive direction of y -axis, the thrust angle δ must be:*

$$-\pi \leq u_2(t) \leq \pi$$

- (b) *If the objective is to make $y(t_f) = 3\text{mi}$, then a state constraint is:*

$$x_3(t_f) = 3$$

Maximization of $x(t_f)$ is the same as minimization of $-x(t_f)$ and a performance measure is:

$$J = -x_1(t_f)$$

- (c) *It is desired to reach the point $x = 500\text{ mi}$, $y = 3\text{ mi}$ in 2.5 min with maximum possible vehicle mass. Then:*

$$x_1(2.5) = 500$$

$$x_3(2.5) = 30$$

and the performance measure is:

$$J = -x_5(t_f)$$

or

$$J = \int_0^{2.5} u_1(t) dt$$

This second form minimizes the thrust function over the time interval given. This could be used to save fuel or minimize consumption (if fuel consumption depends on the thrust) or to maximize vehicle mass.

Example 2.5.2 . Zermelo's problem

Consider a ship traveling from point A to point B in rectangular cartesian coordinates (x, y) , through a region of strong currents as shown in Figure 2.8, (Longuski et al., 2013; Weber, 2000). The ship is assumed to have a constant

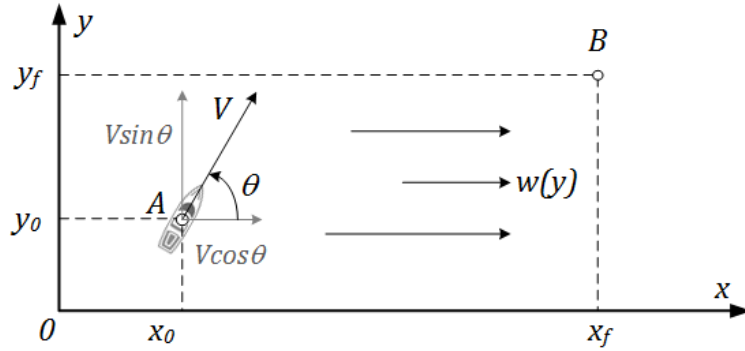


Figure 2.8: Zermelo's problem

speed V with respect to the water, but its heading angle θ can be varied. The velocity of the water current, $w(y)$, is parallel to the x -axis and varies with y . The function w may be a constant value, or a function of time and/or the state variable $y(t)$ and it is given.

The goal is to steer the boat from the initial position $A(x_0, y_0)$ to a given point $B(x_f, y_f)$ in minimum time.

Mathematical model. If the position of the boat at any time moment, t , is $(x(t), y(t))$, the motion of the boat is given by the dynamics:

$$\begin{aligned} \dot{x}(t) &= V \cos \theta(t) + w(y(t)) \\ \dot{y}(t) &= V \sin \theta(t), \quad t \in [0, t_f] \end{aligned}$$

where $\dot{x}(t)$ and $\dot{y}(t)$ are the boat velocities on the x and y -axis, respectively. The velocity of the water current is only added on x -axis, in this particular case.

The system dynamics is described by the state variables $x(t)$ and $y(t)$ and the control signal is the steering angle $\theta(t)$.

Constraints. Since the boat has to travel from the given point $A(x_0, y_0)$ to the given point $B(x_f, y_f)$, the constraints on the states can be formulated as:

$$\begin{aligned}x(0) &= x_0 \\y(0) &= y_0 \\x(t_f) &= x_f \\y(t_f) &= y_f\end{aligned}$$

The control variable $\theta(t)$ is unconstrained.

Performance measure. The goal is to minimize the final time t_f , therefore the performance measure to be minimized is:

$$J = t_f = \int_0^{t_f} dt$$

Summarizing, the optimal control problem can be stated as follows: find the optimal control $\theta^*(t)$ that minimizes the performance measure:

$$J = \int_0^{t_f} dt$$

subject to:

$$\begin{aligned}\dot{x}(t) &= V \cos \theta(t) + w(y(t)) \\ \dot{y}(t) &= V \sin \theta(t)\end{aligned}$$

where $t \in [0, t_f]$ and $x(0) = x_0$, $y(0) = y_0$, $x(t_f) = x_f$, $y(t_f) = y_f$.

Exercise 2.5.1 Formulate the Zermelo problem in case the water velocity is zero $w(y) = 0$ and the coordinates of the initial and final points are $A(0, 0)$ and $B(1, 1)$. The objective is to steer the boat from A to B in minimum time. Write the state differential equations, the constraints and the performance measure.

Exercise 2.5.2 Formulate the Zermelo problem in case the water stream has a constant velocity w in the direction of x -axis and the coordinates of the initial and final points are $A(0,0)$ and $B(0,1)$. The objective is to steer the boat from A to B in minimum time. Write the state differential equations, the constraints and the performance measure.

Exercise 2.5.3 Formulate the Zermelo problem to find the steering angle $\theta^*(t)$ which maximizes the travel distance in the x -direction, in a given time t_f . Assume that the initial position of the boat is $x(0) = y(0) = 0$ and the water stream velocity is linear with respect to y : $w(y(t)) = y(t)$. Write the state differential equations, the constraints and the performance measure.

Example 2.5.3 . Dubins' car

Consider a car modeled as a rigid body moving in the plane (Figure 2.9). The car cannot drive sideways because only the front wheels can be turned. It can move only forward with a fixed linear velocity, V , and simultaneously rotate with a bounded angular velocity, $\dot{\theta}$. The objective is to reach a target in a minimum time.

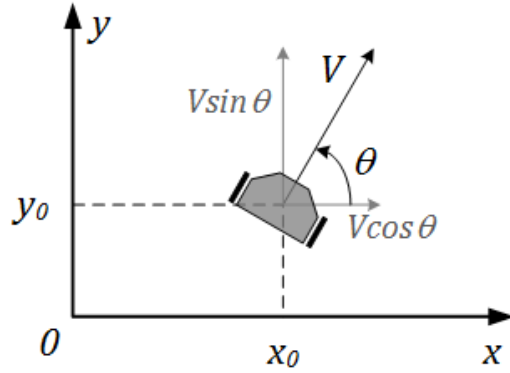


Figure 2.9: Dubins car

The problem has been extensively studied from various points of view in a very large number of academic publications, as for example Dubins (1957); Souères and Laumond (1996); Agrachev (2001); Ma and Castanon (2006); LaValle (2006).

Mathematical model The state of this system is given by the coordinates x , y of the center of the mass of the vehicle and its orientation θ , i.e. the angle made by the velocity vector with the x -axis. The state vector is then, $(x(t), y(t), \theta(t))$ and the control input is the angular velocity,

$u(t) = \dot{\theta}(t)$. Then, the equations describing the motion and orientation of the vehicle can be written as:

$$\begin{aligned}\dot{x}(t) &= V \cos \theta(t) \\ \dot{y}(t) &= V \sin \theta(t) \\ \dot{\theta}(t) &= u(t)\end{aligned}\tag{2.22}$$

If the velocity of the car is assumed equal to 1, the state space model for this system is:

$$\begin{aligned}\dot{x}(t) &= \cos \theta(t) \\ \dot{y}(t) &= \sin \theta(t), \\ \dot{\theta}(t) &= u(t)\end{aligned}\tag{2.23}$$

where $t \in [0, T]$ and the final time T is free.

Constraints Consider the car traveling from an initial position (x_0, y_0) to a final position (x_f, y_f) and having an initial orientation θ_0 . The constraints on the states are then:

$$\begin{aligned}x(0) &= x_0 \\ y(0) &= y_0 \\ \theta(0) &= \theta_0 \\ x(T) &= x_f \\ y(T) &= y_f\end{aligned}\tag{2.24}$$

The final orientation $\theta(T)$ is free. The problem can be also stated in case $\theta(T)$ is fixed.

The control variable $u(t)$ is bounded in the interval $u_{\min} \leq u(t) \leq u_{\max}$.

Performance measure The goal is to minimize the final time T , therefore the performance measure to be minimized is:

$$J = T = \int_0^T dt$$

Summarizing, the optimal control problem can be stated as follows: find the optimal control $u^*(t)$ that minimizes the performance measure:

$$J = \int_0^T dt$$

subject to:

$$\begin{aligned}\dot{x}(t) &= \cos \theta(t) \\ \dot{y}(t) &= \sin \theta(t), \\ \dot{\theta}(t) &= u(t)\end{aligned}\tag{2.25}$$

where $t \in [0, T]$ and $x(0) = x_0$, $y(0) = y_0$, $\theta(0) = \theta_0$, $x(T) = x_f$, $y(T) = y_f$ and $u_{\min} \leq u(t) \leq u_{\max}$.

Note that, if the car velocity is constant ($V = 1$ in this case), the minimum time problem is equivalent to finding a shortest path between the initial and final positions of the vehicle.

Example 2.5.4 Consider the Dubins' car problem where the objective is to drive the car from an initial position with the coordinates $(4, 0)$ to the origin in minimum time. The initial orientation of the car is $\theta_0 = \frac{\pi}{2}$, the control signal is designed to be in the interval $[-1, 1]$. Formulate the optimal control problem.

The optimal control problem can be formulated as follows:
Find a control $u^*(t)$, $|u^*(t)| \leq 1$, that causes the system:

$$\begin{aligned}\dot{x}(t) &= \cos \theta(t) \\ \dot{y}(t) &= \sin \theta(t), \\ \dot{\theta}(t) &= u(t)\end{aligned}$$

to follow an admissible trajectory $(x^*(t), y^*(t), \theta^*(t))$ that minimizes the performance measure:

$$J = \int_0^T dt$$

with the state constraints:

$$\begin{aligned}x(0) &= 4, & x(T) &= 0 \\ y(0) &= 0, & y(T) &= 0 \\ \theta(0) &= \frac{\pi}{2}\end{aligned}$$

Exercise 2.5.4 Formulate the Dubins' car problem if the objective is to drive the car from an initial point $(0, 0)$ to a final point with $x(T) = 4$ in minimum time, if the initial angle is $\theta(0) = \frac{\pi}{2}$ and the control is constrained by $-2 \leq u(t) \leq 2$. The final time T , the final coordinate $y(T)$ and orientation $\theta(T)$ are free.

Exercise 2.5.5 Formulate the Dubins' car problem if the objective is to drive the car as close as possible to a reference trajectory given by the parametric equations $x_{ref}(t) = t$, $y_{ref}(t) = t^2$ starting from an initial point $(0, 0)$, with an initial orientation $\theta(0) = 0$, in a time $T = 10\text{sec}$. The final conditions for the states are free and the control is unconstrained.

Exercise 2.5.6 Consider an inverted pendulum on a cart (Figure 2.10). It is required to maintain the pendulum in an upright position.

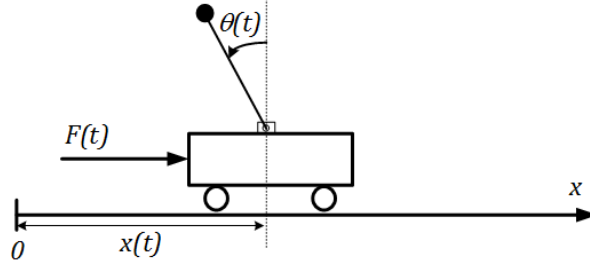


Figure 2.10: Inverted pendulum on a cart

The states for this system are:

- $x_1(t) = x(t)$ - the horizontal displacement of the cart
- $x_2(t) = \dot{x}(t)$ - the linear velocity of the cart
- $x_3(t) = \theta(t)$ - the angular position of the pendulum from the vertical line
- $x_4(t) = \dot{\theta}(t)$ - the angular velocity of the pendulum

The control input is the horizontal force applied to the cart $u(t) = F(t)$. The linearized state equations are, (Naidu, 2002):

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_3(t) + 0.2u(t) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= 10x_3(t) - 0.2u(t)\end{aligned}$$

with $x_1(0) = x_{10}$, $x_2(0) = 0$, $x_3(0) = x_{30}$, $x_4(0) = 0$ and $t \in [0, T]$. Formulate an optimal control problem is the objective is:

- a) to keep the pendulum in the vertical position

- b) to keep the pendulum in the vertical position with a minimum amount of energy expended
- c) to keep the pendulum in the vertical position and to bring the cart to the position $x(T) = 0$ with minimum energy.

Exercise 2.5.7 A first-order system is described by the state equation:

$$\dot{x}(t) = f(x(t), u(t), t)$$

Suggest a performance measure to be minimized and define the constraints (if necessary) if the objective is:

- a) to bring the system from an initial state $x(0) = 0$ to a final state $x(7) = 7$ in minimum time.
- b) to maintain the system state $x(t)$ as close as possible to a desired trajectory $r(t) = (t + 2)e^{-2t}$ in the time interval $[0, 10]$.
- c) to maximize the value of the final state $x(T)$ and minimize the control effort in the time interval $[0, T]$.
- d) to minimize the deviation of the final state $x(T)$ from its desired value 7.

Exercise 2.5.8 Consider a dynamic system described by the state equations:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t)\end{aligned}$$

with the initial conditions $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$ and $t \in [0, 1]$. The objective is to find an optimal control $u^*(t)$ that minimizes a performance measure of the form:

$$a) J = (x_1(1) - 3)^2 + \int_0^1 x_2^2(t) dt$$

$$b) J = -x_2(1) + \int_0^1 (x_1(t) - t)^2 dt$$

Explain in words the objective for each performance measure referring to the states x_1 and x_2 .

Bibliography

- Agrachev, A. (2001). Introduction to optimal control theory. Lectures given at the Summer School on Mathematical Control Theory, Trieste, 3-28 September 2001, http://users.ictp.it/~pub_off/lectures/lns008/Agrachev/Agrachev.pdf.
- Dubins, L. E. (1957). On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. *American Journal of Mathematics*, 79(3):497–516.
- Dutton, K., Thompson, S., and Barraclough, B. (1997). *The Art of Control Engineering*. Addison-Wesley.
- Kirk, D. E. (2004). *Optimal Control Theory. An Introduction*. Dover Publications, Inc.
- LaValle, S. M. (2006). *Planning Algorithms*. Cambridge University Press, 1st edition.
- Longuski, J. M., Guzmán, J. J., and Prussing, J. E. (2013). *Optimal Control with Aerospace Applications. Volume 32 of Space Technology Library*. Springer Science & Business Media.
- Ma, X. and Castanon, D. A. (2006). Receding horizon planning for dubins traveling salesman problems. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 5453–5458.
- Naidu, D. S. (2002). *Optimal Control Systems*. Electrical engineering textbook series. CRC Press.

- Souères, P. and Laumond, J.-P. (1996). Shortest path synthesis for a car-like robot. *IEEE Transactions on Automatic Control*, 41(5):672–688.
- Weber, R. (2000). Optimization and control. online at www.statslab.cam.ac.uk.